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COMMENT

Self-referential decomposition of a class of quadratic irrationals

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Abstract. It has been predicted and numerically shown that the spectrum of the Hamiltonian $H(P)=\Sigma_{n\in\{-\infty,\infty\}}[E(P,n)a_n^\dagger a_n+t(a_{n+1}^\dagger a_n+a_{n-1}^\dagger a_n)]$, in which $E(P,n)=V\cos(P2\pi n)$ and P is an irrational number, has a fractal distribution of eigenstates. Using a self-referential decomposition of a pertinent class of quadratic irrationals, it is shown here that such a conclusion is viable.

In a recent paper, Series (1982) has alluded to the connection existing between fractal geometry and the continued fraction (CF) representation of quadratic irrationals. A CF is a number expressed in the form

$$\mu_0+\frac{1}{\mu_1+}\,\frac{1}{\mu_2+}\,\frac{1}{\mu_3+}\,\frac{1}{\mu_4+}\,\frac{1}{\mu_5+}\,\frac{1}{\mu_6+}\,\frac{1}{\mu_7+}\cdots$$

which is more concisely represented as $\langle\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \ldots\rangle$; we shall take μ_i to be positive integers. The sequence corresponding to a rational number is always terminating, whereas one corresponding to an irrational number continues *ad infinitum*. Our interest lies in the CF representation of irrationals, which are solutions of the quadratic equation. A theorem due to Lagrange (Olds 1963) states that the CF of any quadratic irrational is periodic after a certain stage, e.g.,

$$\sqrt{15}=\langle 3, 1, 6, 1, 6, \ldots\rangle$$

$$(24-\sqrt{15})/17=\langle 1, 5, 2, 3, 2, 3, \ldots\rangle.$$

Such quadratic irrationals have seen varied uses in diverse areas of science, and form the basis of the devil's staircase. Recently, several researchers (e.g., Azbel 1979, Azbel and Rubinstein 1983, Chao 1986) have investigated the spectrum of the Hamiltonian

$$H(P)=\textstyle\sum_{n\in\{-\infty,\infty\}}[E(P,n)a_n^\dagger a_n+t(a_{n+1}^\dagger a_n+a_{n-1}^\dagger a_n)] \tag{1}$$

in which $E(P,n)=V\cos(P2\pi n)$ and $P=\langle 0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \ldots\rangle$ is an irrational number. Azbel (1979) has predicted that the spectrum splits into approximately μ_1 bands, each of which splits into μ_2 sub-bands, each of which splits up into μ_3 sub-sub-bands, and so on. The specific case when $\mu_i=\mu\,\forall i>0$ has been investigated in great detail by Chao (1986) and Azbel's conclusions validated by numerical studies.

Here we follow up on Azbel, as well as on Chao, and show through a CF decomposition of a specific Q that the conclusions of Azbel are viable. Let us consider the irrational

$$Q = \langle \mu_a, \mu_b, a\mu_a, \mu_b, a^2\mu_a, \mu_b, \ldots \rangle \tag{2}$$

whose nth approximant is the rational number

$$Q_n = \langle \mu_a, \mu_b, a\mu_a, \mu_b, a^2\mu_a, \mu_b, \ldots, a^n\mu_a, \mu_b \rangle. \tag{3a}$$

Provided that $\mu_a \geqslant 2$, $\mu_b \geqslant 2$ and $a \geqslant 1$, it can be said (Pringsheim 1899, 1900) that in the limit

$$\lim_{n\to\infty} Q_n = Q. \tag{3b}$$

With some algebraic tedium, it is possible to observe that the higher-order approximants ($n > 0$) can be succinctly put down in terms of Q_0 and an auxiliary sequence $\{q_{1,n}\}$ as

$$Q_n = (Q_0 q_{1,n} + a\mu_a Q_0 + \mu_a \varepsilon_b)(a\mu_a + \varepsilon_b + q_{1,n})^{-1} \qquad n \geqslant 1 \tag{4}$$

in which $Q_0 = \mu_a + \varepsilon_b$, $\mu_b \varepsilon_b = 1$, and

$$q_{1,1} = \varepsilon_b \tag{5a}$$

but

$$q_{1,n} = \varepsilon_b(a^2\mu_a + q_{2,n})(a^2\mu_a + \varepsilon_b + q_{2,n})^{-1} \qquad n \geqslant 2. \tag{5b}$$

In turn, (5b) shows that the auxiliary sequence $\{q_{1,n}\}$ can be expressed in terms of the sequence $\{q_{2,n}\}$, which itself is specified by

$$q_{2,2} = \varepsilon_b \tag{6a}$$

$$q_{2,n} = \varepsilon_b(a^3\mu_a + q_{3,n})(a^3\mu_a + \varepsilon_b + q_{3,n})^{-1} \qquad n \geqslant 3. \tag{6b}$$

Further analysis reveals that

$$q_{3,3} = \varepsilon_b \tag{7a}$$

$$q_{3,n} = \varepsilon_b(a^4\mu_a + q_{4,n})(a^4\mu_a + \varepsilon_b + q_{4,n})^{-1} \qquad n \geqslant 4. \tag{7b}$$

Thus the irrational given in (2) has been shown to be fractally (or self-referentially) built up through the bivariate sequence $\{q_{m,n}\}$ whose elements are specified as

$$q_{n,n} = \varepsilon_b \tag{8a}$$

$$q_{m,n} = 0 \qquad n < m \tag{8b}$$

$$q_{m,n} = \varepsilon_b(a^{m+1}\mu_a + q_{m+1,n})(a^{m+1}\mu_a + \varepsilon_b + q_{m+1,n})^{-1} \qquad n \geqslant m+1. \tag{8c}$$

To be noted here is that should $a = 1$ then the convergence relation (3b) is satisfied more slowly than when $a > 1$.

Several points need to be made here. First, the parameter a assumes a special status, i.e. it imparts a logarithmic scale to the numerators and the denominators of some of the elements of the sequence $\{q_{m,n}\}$, *vide* (8c). Coupled with the fact that Q can be decomposed in terms of all non-zero elements of this sequence, it becomes clear that the spectrum of $H(Q)$ must be self-affine. Also, if one interprets μ_a and ε_b as impedances, then the $q_{m,n}$ of (8c) is nothing but the equivalent impedance of ε_b and $(a^{m+1}\mu_a + q_{m+1,n})$ in parallel.

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